

Nature of Roots (Page 165)

The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Where we take a, b & c as rational)

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$ called *discriminant*.

Case I: If $b^2 - 4ac = 0$

Then roots of the equation are $-\frac{b}{2a}$ and $-\frac{b}{2a}$.

So the roots are real (rational) and repeated equal.

Case II: If $b^2 - 4ac < 0$

Then the roots are complex/imaginary and distinct/unequal.

Case III: If $b^2 - 4ac > 0$

Then the roots are real and distinct/unequal.

However, if $b^2 - 4ac$ is a perfect square then $\sqrt{b^2 - 4ac}$ will be rational and so the roots are rational and unequal. And if $b^2 - 4ac$ is not a perfect square then $\sqrt{b^2 - 4ac}$ will be irrational and so the roots are irrational and unequal.

Question # 1(i)

$$4x^2 + 6x + 1 = 0$$

Here $a = 4$, $b = 6$, $c = 1$

$$\text{Disc.} = b^2 - 4ac$$

$$= (6)^2 - 4(4)(1) = 36 - 16$$

$$= 20 > 0$$

Discriminant is not perfect square therefore the roots are irrational (real) and unequal.

(ii) $x^2 - 5x + 6 = 0$

$a = 1$, $b = -5$, $c = 6$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-5)^2 - 4(1)(6)$$

$$= 25 - 24 = 1 > 0$$

Disc. is perfect square therefore roots are rational (real) and unequal.

(iii) Do yourself as (i)

(iv) $25x^2 - 30x + 9 = 0$

$a = 25$, $b = -30$, $c = 9$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-30)^2 - 4(25)(9)$$

$$= 900 - 900 = 0$$

\therefore roots are rational (real) and equal.

Question # 2(i)

$$x^2 - 2\left(m + \frac{1}{m}\right)x + 3 = 0$$

Here $a = 1$, $b = -2\left(m + \frac{1}{m}\right)$, $c = 3$

$$\text{Disc.} = b^2 - 4ac$$

$$= \left(-2\left(m + \frac{1}{m}\right)\right)^2 - 4(1)(3)$$

$$= 4\left(m^2 + \frac{1}{m^2} + 2\right) - 12$$

$$= 4\left(m^2 + \frac{1}{m^2} + 2 - 3\right)$$

$$= 4\left(m^2 + \frac{1}{m^2} - 1\right)$$

$$= 4\left(m^2 + \frac{1}{m^2} - 2 + 1\right)$$

$$= 4\left(\left(m - \frac{1}{m}\right)^2 + 1\right) > 0$$

Hence roots are real.

Question # 2(ii)

$$(b-a)x^2 + (c-a)x + (a-b) = 0$$

Here $A = b - a$, $B = c - a$, $C = a - b$

$$\text{Disc.} = b^2 - 4ac$$

$$= (c-a)^2 - 4(b-a)(a-b)$$

$$= c^2 + a^2 - 2ca - 4(ab - b^2 - ac + bc)$$

$$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$$

$$= (a^2 + c^2 + 2ac) - 4ab - 4bc + 4b^2$$

$$= (a+c)^2 - 4b(a+c) + (2b)^2$$

$$= (a+c-2b)^2 > 0$$

Hence roots are real.

Question # 3

(i) $(p+q)x^2 - px - qb^2 - 4ac = 0$

Here $a = p + q$, $b = -p$, $c = -q$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-p)^2 - 4(p+q)(-q)$$

$$= p^2 + 4pq + 4q^2$$

$$= (p+2q)^2$$

\therefore the roots are rational.

(ii) $px^2 - (p-q)x - q = 0$

Do yourself

Question # 4

(i) $(m+1)x^2 + 2(m+3)x + m + 8 = 0$

$a = m + 1$, $b = 2(m + 3)$, $c = m + 8$

$$\text{Disc.} = b^2 - 4ac$$

$$= (2(m+3))^2 - 4(m+1)(m+8)$$

$$= 4(m^2 + 6m + 9) - 4(m^2 + 8m + m + 8)$$

$$= 4(m^2 + 6m + 9 - m^2 - 8m - m - 8)$$

$$= 4(-3m + 1)$$

For equal roots, we have

$$\text{Disc.} = 0$$

$$\Rightarrow 4(-3m + 1) = 0$$

$$\Rightarrow -3m + 1 = 0$$

$$\Rightarrow 3m = 1 \Rightarrow \boxed{m = \frac{1}{3}}$$

(ii) & (iii)

Do yourself

Question # 5

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$

$$\Rightarrow x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0$$

Here $A = 1 + m^2$, $B = 2mc$, $C = c^2 - a^2$

So $\text{Disc.} = B^2 - 4AC$

$$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$$

$$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$$

$$= 4(m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2)$$

$$= 4(-c^2 + a^2 + m^2a^2)$$

For equal roots, we have

$$\text{Disc.} = 0$$

$$-c^2 + a^2 + m^2a^2 = 0$$

$$\Rightarrow c^2 = a^2 + m^2a^2$$

$$\Rightarrow c^2 = a^2(1 + m^2)$$

as required.

Question # 6

$$(mx + c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$$

$$\Rightarrow m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

$A = m^2$, $B = 2(mc - 2a)$, $C = c^2$

$\text{Disc.} = B^2 - 4AC$

$$= [2(mc - 2a)]^2 - 4m^2c^2$$

$$= 4(m^2c^2 + 4a^2 - 4amc - m^2c^2)$$

$$= 4(4a^2 - 4amc)$$

For equal roots, we must have

$$\text{Disc.} = 0$$

$$\Rightarrow 4(4a^2 - 4amc) = 0$$

$$\Rightarrow 16a(a - mc) = 0$$

$$\Rightarrow a - mc = 0 \Rightarrow a = mc$$

$$\Rightarrow \frac{a}{m} = c \text{ or } \boxed{c = \frac{a}{m}}$$

Question # 7

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow b^2x^2 + a^2(m^2x^2 + c^2 + 2mcx) - a^2b^2 = 0$$

$$\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Here $A = b^2 + a^2m^2$, $B = 2a^2mc$,

$$C = a^2(c^2 - b^2)$$

$\text{Disc.} = B^2 - 4AC$

$$= (2a^2mc)^2 - 4(b^2 + a^2m^2) \cdot a^2(c^2 - b^2)$$

$$= 4a^4m^2c^2 - 4a^2(c^2b^2 - b^4 + a^2c^2m^2 - a^2b^2m^2)$$

$$= 4a^2(a^2m^2c^2 - c^2b^2 + b^4 - a^2c^2m^2 + a^2b^2m^2)$$

$$= 4a^2(-b^2c^2 + b^4 + a^2b^2m^2)$$

For equal roots we must have

$$\text{Disc.} = 0$$

$$\Rightarrow 4a^2b^2(-c^2 + b^2 + a^2m^2) = 0$$

$$\Rightarrow -c^2 + b^2 + a^2m^2 = 0 \quad \because a \neq 0, b \neq 0$$

$$\Rightarrow c^2 = a^2m^2 + b^2$$

Question # 8

$$(a^2 - ba)x^2 + 2(b^2 - ac)x + c^2 - ab = 0$$

$A = a^2 - bc$, $B = 2(b^2 - ac)$, $C = c^2 - ab$

$\text{Disc.} = B^2 - 4AC$

$$= [2(b^2 - ac)]^2 - 4(a^2 - bc)(c^2 - ab)$$

$$= 4(b^4 + a^2c^2 - 2ab^2c) - 4(a^2c^2 - a^3b + bc^3 - ab^2c)$$

$$= 4(b^4 + a^2c^2 - 2ab^2c - a^2c^2 + a^3b + bc^3 - ab^2c)$$

$$= 4(a^3b + b^4 + bc^3 - 3ab^2c)$$

$$= 4b(a^3 + b^3 + c^3 - 3abc)$$

For equal roots, we must have

$$B^2 - 4AC = 0$$

$$\Rightarrow 4b(a^3 + b^3 + c^3 - 3abc) = 0$$

$$\Rightarrow 4b = 0 \text{ or } a^3 + b^3 + c^3 - 3abc = 0$$

$$\Rightarrow b = 0 \text{ or } a^3 + b^3 + c^3 = 3abc$$

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