Remainder Theorem:-

If a polynomial $f(x)$ of degree $n \geq 1$, n is non negative integer is divided by $(x-a)$ till no x term exists in the remainder Then $f(a)$ is a remainder

PROOF: Suppose a polynomial $f(x)$ is divided by $(x-a)$ Then there exists a unique quotient $q(x)$ and a unique remainder R

Dividend = Divisor × quotient + Remainder i.e

$$f(x) = (x-a)\,q(x) + R$$

putting $x = a$ we get

$$f(a) = (a-a)\,q(a) + R = 0 + R = R$$

Factor Theorem:-

The polynomial $(x-a)$ is a factor of polynomial $f(x)$ if and only if $f(a)=0$ i.e $(x-a)$ is a factor of $f(x)$ if and only if $x=a$ is a root of polynomial equation $f(x)=0$

PROOF: Suppose $q(x)$ is the quotient and R is remainder when a polynomial $f(x)$ is divided by $(x-a)$.

Then by remainder Theorem

$$f(x) = (x-a)\,q(x) + R$$

Since $f(a) = 0 \Rightarrow R = 0$

$\therefore f(x) = (x-a)\,q(x)$

$(x-a)$ is a factor of $f(x)$

Conversely if $(x-a)$ is a factor of $f(x)$ Then

$$R = f(a) = 0$$

which proves the theorem

EXERCISE 4.5

Use The remainder Theorem To find the remainder

1. Let $f(x) = x^2+3x+7$

Sol. $x-a = x+1 \Rightarrow a = -1$

Remainder $= f(a)$ By Remainder Theorem

$R = f(-1) = (-1)^2 + 3(-1) + 7$

$= 1 - 3 + 7 = 5$

2. Let $f(x) = x^3 - x^2 + 5x + 4$

$x-a = x-2 \Rightarrow a = 2$

Remainder $= f(a)$

$R = f(2) = (2)^3 - (2)^2 + 5(2) + 4$

$= 8 - 4 + 10 + 4 = 18$

3. Let $f(x) = 3x^4 + 4x^3 + x - 5$

$x-a = x+1 \Rightarrow a = -1$

$R = f(a) = f(-1) = 3(-1)^4 + 4(-1)^3 - 1 - 5$

$= 3 - 4 - 1 - 5 = -7$

4. $f(x) = x^3 - 2x^2 + 3x + 3$

$x-a = x-3 \Rightarrow a = 3$

$R = f(a) = f(3) = 3^3 - 2(3)^2 + 3(3) + 3$

$= 27 - 18 + 9 + 3 = 21$

Use factor Theorem .....

5. $x-1$, x^2+4x-5

Sol. Let $f(x) = x^2 + 4x - 5$

and $x-a = x-1 \Rightarrow a = 1$

Remainder $= f(a) = f(1)$

$R = (1)^2 + 4(1) - 5 = 5 - 5 = 0$

Hence $(x-1)$ is a factor of $f(x)$ by factor theorem.

6. Let $f(x) = x^3 + x^2 - 7x + 1$

$x-a = x-2 \Rightarrow a = 2$

$R = f(a) = f(2) = 2^3 + 2^2 - 7(2) + 1$

$= 8 + 4 - 14 + 1 = -1 \neq 0$

$\Rightarrow (x-2)$ is not a factor of $f(x)$

7. Let $f(w) = 2w^3 + w^2 - 4w + 7$

$w-a = w+2 \Rightarrow a = -2$

Remainder $= f(\omega) = f(-2)$

$R = 2(-2)^3 + (-2)^2 - 4(-2) + 7$

$= -16 + 4 + 8 + 7 = 3 \neq 0$

$\Rightarrow (\omega + 2)$ is not a factor of $f(\omega)$

8. Let $f(x) = x^n - a^n$ where n is +ve integer

$x - a = x - a \Rightarrow a = a$

$R = f(x) = f(a) = a^n - a^n = 0$

$\Rightarrow (x-a)$ is a factor of $f(x)$

9. Let $f(x) = x^n + a^n$ where n is odd integer

$x - a = x + a \Rightarrow a = -a$

$R = f(x) = f(-a) = (-a)^n + a^n$

$= -a^n + a^n = 0$

$\Rightarrow (x+a)$ is a factor of $f(x)$

10. Let $f(x) = x^4 + 2x^3 + kx^2 + 3$

$k = ?$ $R = 1$

$x - a = x - 2 \Rightarrow a = 2$

$R = f(a) = f(2)$

$1 = (2)^4 + 2(2)^3 + k(2)^2 + 3$

$1 = 16 + 16 + 4k + 3$

$\Rightarrow 4k = -34 \Rightarrow k = -\frac{17}{2}$

11. $k = ?$ $R = 14$

Sol. $f(x) = x^3 + 2x^2 + kx + 4$

$x - a = x - 2 \Rightarrow a = 2$

$R = f(x) = f(a) = f(2)$

$14 = (2)^3 + 2(2)^2 + 2k + 4$

$14 = 8 + 8 + 2k + 4$

$\Rightarrow 2k = -6 \Rightarrow k = -3$

Use Synthetic division....

12. factorize the polynomial

Sol $f(x) = x^3 + 0x^2 - 7x + 6$ $x = 2$

2	1	0	−7	6
	0	2	4	−6
	1	2	−3	0

Quotient $= x^2 + 2x - 3$ $R = 0$

$= x^2 + 3x - x - 3$

$= x(x+3) - 1(x+3)$

$= (x+3)(x-1)$

Hence $x^3 - 7x + 6 = (x-2)(x+3)(x-1)$

13. let $f(x) = x^3 + 0 \cdot x^2 - 28x - 48$ $x = -4$

−4	1	0	−28	−48
	0	−4	16	48
	1	−4	−12	0

Quotient $= x^2 - 4x - 12$

$= x^2 - 6x + 2x - 12$

$= x(x-6) + 2(x-6)$

$= (x-6)(x+2)$

Hence $x^3 - 28x - 48 = (x+4)(x-6)(x+2)$

14. Let $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

$x = 2, \quad x = -3$

2	2	7	−4	−27	−18
	0	4	22	36	18
−3	2	11	18	7	0
	0	−6	−15	−9	
	2	5	3	0	

Quotient $= 2x^2 + 5x + 3$

$= 2x^2 + 2x + 3x + 3$

$= 2x(x+1) + 3(x+1)$

$= (x+1)(2x+3)$

Hence $2x^4 + 7x^3 - 4x^2 - 27x - 18$

$= (x-2)(x+3)(x+1)(2x+3)$

15. Use Synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of $f(x) = x^3 + px^2 + qx + 6$

Sol $x - a = x + 1 \Rightarrow a = -1$

$x - a = x - 2 \Rightarrow a = 2$

By Synthetic division

−1	1	p	q	6
	0	−1	−p+1	q−p+1
2	1	p−1	q−p+1	q+p+5
	0	2	2p+2	
	1	p+1	q+p+3	

Since $(x+1)$ and $(x-2)$ are the factors of $f(x)$

Then Remainder $= 0$

$-q+p+5=0$ ---(1)

$q+p+3=0$ ----(2)

$2p+8=0 \Rightarrow p=-4$

put $p=-4$ in (2)

$\Rightarrow q-4+3=0 \Rightarrow q=1$

$p=-4 \quad q=1$

16. Find the values of a and b if -2 and 2 are the roots of $f(x)=x^3-4x^2+ax+b$

Sol. $a=-2 \quad a=2$

-2	1	-4	a	b
	0	-2	12	$-2a-24$
2	1	-6	$a+12$	$b-2a-24$
	0	2	-8	
	1	-4	$a+4$	

Since $-2, 2$ are the roots of $f(x)$ then Remainder $=0$

$a+4=0 \Rightarrow a=-4$

$b-2a-24=0$

$\Rightarrow b-2(-4)-24=0$

$\Rightarrow b+8-24=0$

$\Rightarrow b-16=0 \Rightarrow b=16$

$a=-4 \quad b=16$

Relation Between Roots & Coefficients of Quadratic Eq

Quadratic eq $ax^2+bx+c=0$

formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

let α, β be the roots of the eq

$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$

$=\frac{-2b}{2a}=-\frac{b}{a}=$ Sum of roots

$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$

$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{2+a^2}$

$P=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$

Product of roots $=\frac{c}{a}$

Formation of an Eq whose Roots are given:

If α, β be the roots of the quadratic equation

Then $x=\alpha \quad x=\beta$

$\Rightarrow x-\alpha=0 \quad x-\beta=0$

Equation becomes

$(x-\alpha)(x-\beta)=0$

$\Rightarrow x^2-\beta x-\alpha x+\alpha\beta=0$

$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$

$\Rightarrow x^2-Sx+P=0$