

1. Evaluate the Determinant

$$i) \begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$$

$$= 5 \begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= 5(-2+3) + 2(6-6) - 4(3-2)$$

$$= 5(1) + 2(0) - 4(1)$$

$$= 5 - 4 = 1$$

$$ii) \begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= 5 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} + (-3) \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= 5(2-1) - 2(-6+2) - 3(3-2)$$

$$= 5(1) - 2(-4) - 3(1)$$

$$= 5 + 8 - 3 = 10$$

$$iii) \begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 1(18-20) - 2(-6+8) - 3(-5+6)$$

$$= 1(-2) - 2(2) - 3(1)$$

$$= -2 - 4 - 3 = -9$$

$$iv) \begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$$

$$= \begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix} \begin{matrix} C_1 + (C_2 + C_3) \\ \\ \end{matrix}$$

$$= 3a \begin{vmatrix} 1 & a-l & a \\ 1 & a+l & a-l \\ 1 & a & a+l \end{vmatrix}$$

$$= 3a \begin{vmatrix} 1 & a-l & a \\ 0 & 2l & -l \\ 0 & l & l \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ \end{matrix}$$

Expand by  $C_1$

$$= 3a [1(2l^2 + l^2) - 0 + 0]$$

$$= 3a(3l^2) = 9al^2$$

$$vi) \begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$$

$$\text{Sol. } abc \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= abc \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -3 & -1 & 2 \end{vmatrix}$$

Expand by  $R_1$

$$= abc [0 - 0 + 1 \begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix}]$$

$$= abc [1(1+3)]$$

$$= 4abc$$

2. Without expansion show that

$$i) \begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\text{Sol. L.H.S. } \begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ \end{matrix}$$

$$= 0 \quad (\because C_2, C_3 \text{ are same})$$

$$= \text{R.H.S.}$$

$$ii) \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$$

$$\text{Sol. L.H.S. } \begin{vmatrix} -1 & 3 & -1 \\ 0 & 1 & 0 \\ 5 & -3 & 5 \end{vmatrix} \begin{matrix} C_1 - C_2 \\ \\ \end{matrix}$$

$$= 0 \quad (\because C_1 = C_3)$$

$$= \text{R.H.S.}$$

$$iii) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Sol. L.H.S  $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_2 \end{matrix}$   
 $= 0 \quad (\because R_2 = R_3)$   
 $= R.H.S$

3. i) Show that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

Sol L.H.S Expand by  $C_3$

$$\begin{aligned} &= (a_{13} + \alpha_{13}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - (\alpha_{23} + \alpha_{23}) \\ &\quad \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + (a_{33} + \alpha_{33}) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &+ \alpha_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix} \end{aligned}$$

ii)  $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

L.H.S Taking 3 common from  $R_2$

$$= 3 \begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1 \end{vmatrix}$$

Taking 3 common from  $C_2$

$$= 9 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix} = R.H.S$$

iii)  $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l(3a+l)$

L.H.S  $\begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \begin{matrix} R_1 + \\ (R_2 + R_3) \end{matrix}$

$$= (3a+l) \begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$= (3a+l) \begin{vmatrix} 1 & 0 & 0 \\ a & l & 0 \\ a & 0 & l \end{vmatrix} \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$\begin{aligned} &= (3a+l) \begin{vmatrix} 1 & l & 0 \\ 0 & l & 0 \\ 0 & 0 & 0 \end{vmatrix} = -0 + 0 \\ &= (3a+l) [1(l^2 - 0) - 0] \\ &= (3a+l) l^2 = R.H.S \end{aligned}$$

iv)  $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

L.H.S Multiplying  $C_1$  by  $x$ ,  $C_2$  by  $y$ ,  $C_3$  by  $z$  and dividend Det by  $xyz$

$$= \frac{1}{xyz} \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Taking  $xyz$  common from  $R_3$

$$= \frac{xyz}{xyz} \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Interchanging  $R_2$  and  $R_3$

$$= (-1) \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Interchanging  $R_1$  and  $R_2$

$$= (-1)(-1) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = R.H.S$$

v)  $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol. L.H.S  $\begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \begin{matrix} R_1 - (R_2 + R_3) \end{matrix}$

$$= -2 \begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{aligned}
 &= -2 \left( 0 - c \begin{vmatrix} b & 0 \\ c & a \end{vmatrix} + b \begin{vmatrix} b & a \\ c & 0 \end{vmatrix} \right) \\
 &= -2 \left[ -c(ab - 0) + b(0 - ac) \right] \\
 &= -2 \left[ -abc - abc \right] \\
 &= -2 \left[ -2abc \right] = 4abc = R.H.S
 \end{aligned}$$

**vii)**  $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$

Sol. Expand by  $R_1$

$$\begin{aligned}
 &= b \begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1) \begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a \begin{vmatrix} a & b \\ 1 & a \end{vmatrix} \\
 &= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b) \\
 &= b^3 + ab + a^3 - ab \\
 &= a^3 + b^3 = R.H.S
 \end{aligned}$$

**viii)**  $\begin{vmatrix} r \cos \phi & 1 & -\sin \phi \\ 0 & r & 0 \\ r \sin \phi & 0 & \cos \phi \end{vmatrix} = r^2$

Sol. L.H.S

$$\begin{aligned}
 &= r \cos \phi \begin{vmatrix} r & 0 \\ 0 & \cos \phi \end{vmatrix} - 1 \begin{vmatrix} r \sin \phi & 0 \\ r \sin \phi & \cos \phi \end{vmatrix} - \sin \phi \begin{vmatrix} r \sin \phi & r \\ r \sin \phi & 0 \end{vmatrix} \\
 &= r \cos \phi (r \cos \phi) - 0 - \sin \phi (0 - r^2 \sin \phi) \\
 &= r^2 \cos^2 \phi + r^2 \sin^2 \phi \\
 &= r^2 (1) = r^2 = R.H.S
 \end{aligned}$$

**viii)**  $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & b+c & a+b \\ a+b+c & c+a & b+c \\ a+b+c & a+b & c+a \end{vmatrix} \quad C_1 + C_2$$

Taking  $(a+b+c)$  common from  $C_1$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expand by  $C_1$

$$\begin{aligned}
 &= (a+b+c) \left( 1 \begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0 \right) \\
 &= (a+b+c) \left[ (a-b)(c-b) - (a-c)(c-a) \right] \\
 &= (a+b+c) \left[ ac - ab - bc + b^2 - (ac - a^2 - c^2 + ac) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) \\
 &= a^3 + b^3 + c^3 - 3abc = R.H.S
 \end{aligned}$$

**ix)**  $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2 (a+b+c+\lambda)$

Sol. L.H.S

$$\begin{vmatrix} a+b+c+\lambda & b & c \\ a+b+c+\lambda & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \quad C_1 + (C_2 + C_3)$$

$$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix}$$

$$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expand from  $C_1$

$$\begin{aligned}
 &= (a+b+c+\lambda) \left( 1 \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0 \right) \\
 &= (a+b+c+\lambda) \left( 1(\lambda^2 - 0) \right) \\
 &= (a+b+c+\lambda) \lambda^2 = R.H.S
 \end{aligned}$$

**x)**  $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol. L.H.S

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2 - b^2 & b^2 - c^2 & c^2 \end{vmatrix} \quad \begin{matrix} C_1 - C_2 \\ C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b+c)(b-c) & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \left( 0 - 0 + 1 \begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix} \right)$$

$$= (a-b)(b-c) \left( 1(b+c - a - b) \right)$$

$$= (a-b)(b-c)(c-a)$$

$$= R.H.S$$

**xi)**

$$\text{L.H.S} \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix} \quad C_1 + C_2$$

$$= (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

by  $R_1 - R_2, R_2 - R_3$

$$= (a+b+c) \begin{vmatrix} 0 & a-b & a^2 - b^2 \\ 0 & b-c & b^2 - c^2 \\ 1 & c & c^2 \end{vmatrix}$$

Expand by  $C_1$

$$= (a+b+c) \left[ 0 - 0 + 1 \begin{vmatrix} a-b & (a-b)(a+b) \\ b-c & (b-c)(b+c) \end{vmatrix} \right]$$

$$= (a+b+c)(a-b)(b-c) \begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$$

$$= (a+b+c)(a-b)(b-c)(b+c-a-b)$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

$$= \text{R.H.S}$$

**4.** If  $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{pmatrix}$   $B = \begin{pmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{pmatrix}$

i)  $A_{12}, A_{22}, A_{32}, |A|$

Sol  $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} = (-1)^3 (0 - 0) = 0$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = (-1)^4 (1 - 6) = 1(-5) = -5$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)^5 (0 + 0) = 0$$

$$|A| = a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32}$$

$$= 2(0) + (-2)(-5) + (-2)(0)$$

$$= 0 + 10 - 0 = 10$$

ii)  $B_{21}, B_{22}, B_{23}, |B|$

$$B_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix} = (-1)^3 (4 - 5) = -1(-1) = 1$$

$$B_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix} = (-1)^4 (-10 + 10) = 0$$

$$B_{23} = (-1)^{2+3} \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = (-1)^5 (5 - 4) = -1(1) = -1$$

$$|B| = b_{21} B_{21} + b_{22} B_{22} + b_{23} B_{23}$$

$$= 3(1) + (-1)(0) + 4(-1)$$

$$= 3 - 0 - 4 = -1$$

**5.** Without expansion verify that

i)  $\begin{vmatrix} \alpha & \beta + \gamma & 1 \\ \beta & \alpha + \gamma & 1 \\ \gamma & \alpha + \beta & 1 \end{vmatrix}$

Sol.  $\text{L.H.S} = \begin{vmatrix} \alpha + \beta + \gamma & \beta + \gamma & 1 \\ \alpha + \beta + \gamma & \alpha + \gamma & 1 \\ \alpha + \beta + \gamma & \alpha + \beta & 1 \end{vmatrix} \quad C_1 + C_2$

$$= (\alpha + \beta + \gamma) \begin{vmatrix} 1 & \beta + \gamma & 1 \\ 1 & \alpha + \gamma & 1 \\ 1 & \alpha + \beta & 1 \end{vmatrix}$$

$$= (\alpha + \beta + \gamma)(0) \quad (\because C_1 = C_3)$$

$$= 0 = \text{R.H.S}$$

ii)  $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 6 & 9x \end{vmatrix} = 0$

Sol.  $\text{L.H.S} = (3x)$  Common from  $C_3$ .

$$= 3x \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$$= 3x(0) \quad (\because C_1 = C_3)$$

$$= 0 = \text{R.H.S}$$

iii)  $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

Sol. Multiplying  $C_3$  by  $abc$  and dividing Det by  $abc$ .

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & abc \frac{a}{bc} \\ 1 & b^2 & abc \frac{b}{ca} \\ 1 & c^2 & abc \frac{c}{ab} \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc} (0) \quad (\because C_2 = C_3)$$

$$= 0 = \text{R.H.S}$$

$$iv) \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Sol. L.H.S  $C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \because \text{all elements of } C_1 \text{ are zero}$$

$$= 0 = R.H.S$$

$$v) \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

Sol. L.H.S Multiplying  $R_2$  by  $abc$  and dividing Det by  $abc$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ abc(\frac{1}{a}) & abc(\frac{1}{b}) & abc(\frac{1}{c}) \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ac & ab \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc} (0) \because R_1 = R_2$$

$$= 0 = R.H.S$$

$$vii) \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

Sol. L.H.S Multiplying  $R_1$  by  $l$

$R_2$  by  $m$ ,  $R_3$  by  $n$  and dividing det by  $lmn$

$$= \frac{1}{lmn} \begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$$

$$= \frac{1}{lmn} \begin{vmatrix} lmn & 1 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$$

$$= R.H.S$$

$$viii) \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$$

Sol L.H.S

$$= 2 \begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 2bc \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2bc(0) = 0 \text{ R.H.S } (\because R_2 = R_3)$$

$$ix) \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

Sol. R.H.S By adding  $C_3$

$$\begin{vmatrix} 7 & 2 & 7+(-1) \\ 6 & 3 & 5+(-3) \\ -3 & 5 & -3+4 \end{vmatrix}$$

$$\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = L.H.S$$

$$x) \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$$

Sol. Multiplying  $C_1$  by  $c$ ,  $C_2$  by  $b$ ,  $C_3$  by  $a$  and dividing Det by  $abc$

$$= \frac{1}{abc} \begin{vmatrix} -ac & 0 & ac \\ 0 & ab & -ab \\ bc & -bc & 0 \end{vmatrix}$$

$$C_1 + (C_2 + C_3)$$

$$= \frac{1}{abc} \begin{vmatrix} a & 0 & ac \\ 0 & ab & -ab \\ 0 & -bc & 0 \end{vmatrix}$$

$$= \frac{1}{abc} (0) \because \text{All entries of } C_1 \text{ are zero}$$

$$= R.H.S$$

6. Find  $x$  if

Sol. (i) 
$$\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$$

$$3 \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x \begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$$

$$3(0-4) - 1(0-4x) + x(-1-3x) = -30$$

$$-12 + 4x - x - 3x^2 + 30 = 0$$

$$-3x^2 + 3x + 18 = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x^2 - 3x + 2x - 6 = 0$$

$$\Rightarrow x(x-3) + 2(x-3) = 0$$

$$\Rightarrow (x-3)(x+2) = 0$$

$$\Rightarrow x-3=0 \quad x+2=0$$

$$\Rightarrow x=3 \quad x=-2$$

ii) 
$$\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$$

$$1 \begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1) \begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3 \begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$$

$$x^2 + x + 4 - (x-1)(-x-4) + 3(2-2x-2) = 0$$

$$x^2 + x + 4 + (x-1)(x+4) - 6x = 0$$

$$x^2 + x + 4 + x^2 - x + 4x - 4 - 6x = 0$$

$$2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$$

$$2x = 0 \Rightarrow x = 0$$

$$x-1 = 0 \Rightarrow x = 1$$

iii) 
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$$

Sol. 
$$1 \begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1 \begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$$x^2 - 12 - 2(2x-6) + 12 - 3x = 0$$

$$\Rightarrow x^2 - 12 - 4x + 12 + 12 - 3x = 0$$

$$\Rightarrow x^2 - 7x + 12 = 0$$

$$\Rightarrow (x-4)(x-3) = 0$$

$$\Rightarrow x-4=0 \quad x-3=0$$

$$x=4 \quad x=3$$

7. Evaluate

(i) 
$$\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

Sol.

$$\begin{vmatrix} 0 & -2 & 11 & -8 \\ 0 & 1 & 6 & -7 \\ 1 & 2 & -3 & 5 \\ 0 & -7 & 10 & 14 \end{vmatrix} \begin{array}{l} R_1 - 3R_3 \\ R_2 - 2R_3 \\ R_4 - 4R_3 \end{array}$$

Expand by  $C_1$

$$= 1(-1) \begin{vmatrix} -2 & 11 & -8 \\ 1 & 6 & -7 \\ -7 & 10 & -14 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 23 & -22 \\ 1 & 6 & -7 \\ 0 & 52 & -63 \end{vmatrix} \begin{array}{l} R_1 + 2R_2 \\ R_3 + 7R_2 \end{array}$$

$$= - \begin{vmatrix} 23 & -22 \\ 52 & -63 \end{vmatrix}$$

$$= -(-23 \times 63 + 22 \times 52)$$

$$= -(-1449 + 1144) = 305$$

ii) 
$$\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -6 & 2 & 3 \\ 7 & 5 & -1 & 5 \\ -1 & -13 & 2 & 0 \end{vmatrix} \begin{array}{l} C_1 - 2C_3 \\ C_2 - 3C_3 \\ C_4 + C_3 \end{array}$$

Expand by  $R_1$

$$= 1(-1) \begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 3 \\ 7 & 15 & 5 \\ -1 & -13 & 0 \end{vmatrix} \quad C_2 + 2C_3$$

$$= 3 \begin{vmatrix} 7 & 15 \\ -1 & -13 \end{vmatrix} = 3(-91 + 15)$$

$$= 3(-76) = -228$$

iii) 
$$\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 6 & -15 & -3 & 2 \\ 12 & -2 & -2 & 1 \\ -5 & +9 & 9 & -1 \end{vmatrix} \begin{array}{l} C_1 + 3C_4 \\ C_2 - 9C_4 \\ C_3 - C_4 \end{array}$$

$$= 1(-1) \begin{vmatrix} 1+4 & 6 & -15 & -3 \\ 12 & -2 & -2 & -2 \\ -5 & 9 & 9 & 9 \end{vmatrix}$$

$$= (-1) 3 \times 2 \begin{vmatrix} 2 & -5 & -1 \\ 6 & -1 & -1 \\ -5 & 9 & 9 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 0 & 0 & -1 \\ 4 & 4 & -1 \\ -1 & -1 & 0 \end{vmatrix} \quad \begin{array}{l} C_1 + 2C_2 \\ C_2 - 5C_3 \end{array}$$

$$= -6(-1) \begin{vmatrix} 4 & 4 \\ -1 & -1 \end{vmatrix}$$

$$= 6(-4 + 4) = 0$$

8. Show that

$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$$

Sol.  $R_1 + (R_2 + R_3 + R_4)$

$$= \begin{vmatrix} x+3 & x+3 & x+3 & x+3 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x-1 & 0 & 0 \\ 1 & 0 & x-1 & 0 \\ 1 & 0 & 0 & x-1 \end{vmatrix} \quad \begin{array}{l} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{array}$$

Expand by  $R_1$

$$= (x+3) \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$$= (x+3)(x-1) \begin{vmatrix} x-1 & 0 \\ 0 & x-1 \end{vmatrix}$$

$$= (x+3)(x-1)((x-1)^2 - 0)$$

$$= (x+3)(x-1)^3 = \text{R.H.S}$$

9. Find  $|AA^t|$ ,  $|A^tA|$

i) if  $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$   $A^t = \begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

$$AA^t = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 5 & 14 \end{pmatrix}$$

$$|AA^t| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix} = 196 - 25 = 171$$

$$A^tA = \begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{pmatrix}$$

$$|A^tA| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

$$= 13 \begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8 \begin{vmatrix} 8 & 1 \\ 3 & 10 \end{vmatrix} + 3 \begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 637 - 616 - 21 = 0$$

ii)  $AA^t = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{pmatrix}$$

$$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 8 & 7 & 5 & 13 \end{vmatrix}$$

$$= \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 1 & -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 35 & 7 & -7 \\ 0 & 15 & 3 & -3 \\ 0 & 10 & 2 & -2 \\ 1 & -1 & 0 & 1 \end{vmatrix} \quad \begin{array}{l} R_1 - 25R_4 \\ R_2 - 10R_4 \\ R_3 - 7R_4 \end{array}$$

Expand by  $C_1$

$$= -1 \begin{vmatrix} 35 & 7 & -7 \\ 5 & 3 & -3 \\ 0 & 2 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 35 & 7 & 7 \\ 5 & 3 & 3 \\ 0 & 2 & 2 \end{vmatrix} = 0 \text{ (R.H.S)} \\ (\because C_2 = C_3)$$

$${}^t A A = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{pmatrix}$$

$$|{}^t A A| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix} = 18 \times 27 - 21 \times 21 \\ = 486 - 441 = 45.$$

10. If  $A$  is a square matrix of order 3. Then show that

$$|KA| = K^3 |A|$$

Sol. Let  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$KA = K \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$$

$$= K \cdot K \cdot K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|KA| = K^3 |A|$$

11. Find the values of  $\lambda$  if  $A$  and  $B$  are singular

(i)  $|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix}$

$$= 4(3-18) - \lambda(7-12) + 3(21-6)$$

$$= -60 + 5\lambda + 45 = 5\lambda - 15$$

For singular  $|A| = 0$

$$5\lambda - 15 = 0 \Rightarrow \lambda = 3$$

20

(ii)  $|B| = \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & -1 \\ 2 & \lambda & 1 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 1 \\ -7 & 2 & 4 & 1 \\ 2-5\lambda & \lambda & -1-2\lambda & 3 \end{vmatrix} \begin{matrix} C_1 - 5C_2 \\ C_3 - 2C_2 \end{matrix}$$

$$= 1(-1)^{1+2} \begin{vmatrix} -2 & 1 & 1 \\ -7 & -4 & 1 \\ 2-5\lambda & -1-2\lambda & 3 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 0 & 0 & 1 \\ -5 & -5 & 1 \\ 8-5\lambda & -4-2\lambda & 3 \end{vmatrix} \begin{matrix} C_1 + 2C_3 \\ C_2 - C_3 \end{matrix}$$

$$= -1 \cdot 1 \begin{vmatrix} -5 & -5 \\ 8-5\lambda & -4-2\lambda \end{vmatrix}$$

$$= - (20 + 10\lambda + 40 - 25\lambda)$$

$$= 15\lambda - 60$$

For singular  $|B| = 0$

$$15\lambda - 60 = 0 \Rightarrow \lambda = 4$$

12

(i)  $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$

$$= 1(4-2) - 0 + 3(6-0)$$

$$= 6 + 18 = 24 \neq 0$$

$A$  is non singular

(ii)  $|B| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$

$$= 2(5+0) - 3(5-0) - 1(-3-2)$$

$$= 10 - 15 + 5 = 0$$

$B$  is singular

(iii)  $|C| = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -3 & -2 \\ 2 & 1 & -3 & 4 \\ 3 & -4 & -3 & 7 \end{vmatrix} \begin{matrix} C_2 - C_1 \\ C_3 - 2C_1 \\ C_4 + C_1 \end{matrix}$$



$$= \begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 6 \\ -4 & -15 & -1 \end{vmatrix} \begin{matrix} C_2+3C_1 \\ C_3+2C_1 \end{matrix}$$

$$= 1(0+90) = 90 \neq 0$$

C is non singular

13.

$$A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

Show that  $A^{-1}A = I_3$

$$\text{Sol. } |A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

$$\text{Sol. Cofactors of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} = (-1)^2(-1+12) = 11$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = (-1)^3(1-6) = -(-5) = 5$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -4 \\ 2 & -4 \end{vmatrix} = (-1)^4(-4+2) = 1(-2) = -2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} = (-1)^3(1+0) = -1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = (-1)^4(2-0) = 2$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & -4 \end{vmatrix} = (-1)^5(-8-2) = 10$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = (-1)^4(3+0) = 3$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = (-1)^5(6-0) = -6$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^6(-2-1) = -3$$

$$\text{Cofactors of } A = \begin{bmatrix} 11 & 5 & -2 \\ -1 & 2 & 10 \\ 3 & -6 & -3 \end{bmatrix}$$

$$\text{Adj } A = (\text{Cofactors of } A)^t = \begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

$$= 2(-1+12) - 1(1-6) + 0$$

$$= 22 + 5 = 27$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$= \frac{1}{27} \begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}$$

$$A^{-1}A = \frac{1}{27} \begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ -4 & -3 & 7 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 22-1+5 & 11-1-1 & 0-3+3 \\ 10+2-12 & 5-2+24 & 0+6-6 \\ -4+10-6 & -2-10+12 & 0+30-3 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

R.H.S

14. Verify that  $(AB)^{-1} = B^{-1}A^{-1}$

$$i) A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\text{Sol. } AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+8 & 1-2 \\ 3+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix} = -5+3 = -2$$

$$(AB)^{-1} = \frac{1}{|AB|} \text{Adj } AB = \frac{1}{-2} \begin{bmatrix} -1 & +1 \\ -3 & 5 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix} \quad \text{--- (i)}$$

R.H.S

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0+2 = 2$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = 3-4 = -1$$

$$B^{-1} = \frac{1}{|B|} \text{Adj } B = \frac{1}{-1} \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{+2} \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0+1 & -2+1 \\ 0+3 & -8+3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix}$$

$$ii) \text{ \& } \text{--- (ii)} \quad (AB)^{-1} = B^{-1}A^{-1}$$

ii)  $A = \begin{pmatrix} 5 & 1 \\ 2 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} 5 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 20+2 & 15+1 \\ 8+4 & 6+2 \end{pmatrix}$

$AB = \begin{pmatrix} 22 & 16 \\ 12 & 8 \end{pmatrix}$

$|AB| = \begin{vmatrix} 22 & 16 \\ 12 & 8 \end{vmatrix} = 176 - 192 = -16$

$(AB)^{-1} = \frac{1}{|AB|} \text{Adj} AB = \frac{1}{-16} \begin{pmatrix} 8 & -16 \\ -12 & 22 \end{pmatrix}$

For R.H.S  $B^{-1}A^{-1}$

$|A| = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 10 - 2 = 8$

$A^{-1} = \frac{1}{|A|} \text{Adj} A = \frac{1}{8} \begin{pmatrix} 2 & -1 \\ -2 & 5 \end{pmatrix}$

$|B| = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4 - 6 = -2$

$B^{-1} = \frac{1}{|B|} \text{Adj} B = \frac{1}{-2} \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

$B^{-1}A^{-1} = \frac{1}{-16} \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -2 & 5 \end{pmatrix}$

$= \frac{1}{-16} \begin{pmatrix} 2+6 & -1-15 \\ -4-8 & 2+20 \end{pmatrix} = \frac{1}{-16} \begin{pmatrix} 8 & -16 \\ -12 & 22 \end{pmatrix}$

L.H.S = R.H.S

15. Verify that

$(AB)^t = B^t A^t$  if

Sol.  $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$   $B = \begin{pmatrix} 1 & 1 \\ 3 & -1 \\ 0 & 2 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1-3+0 & 1-2-2 \\ 0+9+0 & 0+6-1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 9 & 5 \end{pmatrix}$

$(AB)^t = \begin{pmatrix} -2 & 9 \\ -3 & 5 \end{pmatrix}$

R.H.S

$B^t A^t = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1-3+0 & 0+9+0 \\ 1-2-2 & 0+6-1 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -3 & 5 \end{pmatrix}$

L.H.S = R.H.S

16. If  $A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$

verify that  $(A^{-1})^t = (A^t)^{-1}$

Sol. L.H.S

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5$

$A^{-1} = \frac{1}{|A|} \text{Adj} A = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$

$(A^{-1})^t = \frac{1}{5} \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$  --- (i)

For R.H.S

$A^t = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$   $|A^t| = 2 + 3 = 5$

$(A^t)^{-1} = \frac{1}{|A^t|} \text{Adj} A^t = \frac{1}{5} \begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix}$  --- (ii)

L.H.S = R.H.S

17. If A and B are non singular

i)  $(AB)^{-1} = B^{-1}A^{-1}$

Sol. Let  $(AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

Now  $(B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Thus  $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$

It shows that AB is the inverse of  $B^{-1}A^{-1}$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

ii)  $(A^{-1})^{-1} = A$

Sol. Since  $A^{-1}A = I$

and  $AA^{-1} = I$

It shows that  $A^{-1}$  be the inverse of A

$(A^{-1})^{-1} = A$

\*\*\*\*\*