

Conjugate of a Complex Number.

For a complex number $z = x + iy$, the complex number $\bar{z} = \overline{x + iy} = x - iy$ is called conjugate of z .

Here z and \bar{z} have same real part but their imaginary parts differ in sign, so z and \bar{z} are called conjugate of each other.

Thus $5 + 4i$ and $5 - 4i$ are conjugate complex numbers.

Note: If $z = \bar{z}$, then z is called self conjugate.

Since for $a \in \mathbb{R}$

$$\bar{a} = a$$

\Rightarrow Every real number is self conjugate.

Powers of i

$$i^2 = -1 \quad \because i = \sqrt{-1}$$

$$i^3 = (i^2) \cdot i = (-1) \cdot i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^5 = (i^2)^2 \cdot i = (-1)^2 \cdot i = 1 \cdot i = i$$

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

$$i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = (-1) \cdot i = -i$$

$$i^8 = (i^2)^4 = (-1)^4 = 1$$

and so on.

\Rightarrow Powers of i should be

expressed in terms of powers of i^2 .

EXERCISE 1.2

① Verify the addition properties of Complex numbers.

Solution:

i) Closure Property

For $(a, b), (c, d) \in \mathbb{C}$

$$(a, b) + (c, d) = (a+c, b+d) \in \mathbb{C}$$

ii) Associative Property

For $(a, b), (c, d), (e, f) \in \mathbb{C}$

$$((a, b) + (c, d)) + (e, f) = (a+c, b+d) + (e, f)$$

$$= ((a+c) + e, (b+d) + f)$$

$$= (a + (c+e), b + (d+f))$$

\because '+' is associative in \mathbb{R}

$$= (a, b) + (c+e, d+f)$$

$$= (a, b) + [(c, d) + (e, f)]$$

iii) Additive Identity

$\forall (a, b) \in \mathbb{C}$ we have $(0, 0) \in \mathbb{C}$ such that

$$(a, b) + (0, 0) = (a+0, b+0) = (a, b)$$

$\Rightarrow (0, 0)$ is additive identity in \mathbb{C}

iv) Additive Inverse

$\forall (a, b) \in \mathbb{C}$ there is $(-a, -b) \in \mathbb{C}$ such that

$$(a, b) + (-a, -b) = (a-a, b-b) = (0, 0)$$

$\Rightarrow (a, b)$ and $(-a, -b)$ are additive inverse of each other.

v) Commutative Property

$\forall (a, b), (c, d) \in \mathbb{C}$

The set $C = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$ is called set of complex numbers.

Operations on Complex Numbers:

Let $z_1 = (x_1, y_1) = x_1 + iy_1$

$z_2 = (x_2, y_2) = x_2 + iy_2$

i) Addition:

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\ &= x_1 + iy_1 + x_2 + iy_2 \\ &= x_1 + x_2 + iy_1 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_1 + x_2, y_1 + y_2) \end{aligned}$$

ii) Subtraction:

$$\begin{aligned} z_1 - z_2 &= (x_1, y_1) - (x_2, y_2) \\ &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= x_1 + iy_1 - x_2 - iy_2 \\ &= x_1 - x_2 + iy_1 - iy_2 \\ &= (x_1 - x_2) + i(y_1 - y_2) \\ &= (x_1 - x_2, y_1 - y_2) \end{aligned}$$

iii) Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1, y_1) \cdot (x_2, y_2) \\ &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + (-1) y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \\ &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$

iv) Division:

$$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\begin{aligned} &= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1 x_2 - iy_1 y_2 + iy_1 x_2 - i^2 y_1 y_2}{x_2^2 - i^2 y_2^2} \\ &= \frac{x_1 x_2 - iy_1 y_2 + iy_1 x_2 - (-1) y_1 y_2}{x_2^2 - (-1) y_2^2} \\ &= \frac{x_1 x_2 - iy_1 y_2 + iy_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \\ &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right) + i \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right) + i \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \\ &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \end{aligned}$$

v) Equality:

$$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$$

vi) Scalar Multiplication

If $z = (x, y) = x + iy$ be a complex number and k is any real constant then

$$\begin{aligned} kz &= k(x, y) = k(x + iy) \\ &= kx + ik y \\ &= (kx, ky) \end{aligned}$$

Note that Every real number a can be written as $a = a + i \cdot 0$
 \therefore Set of real numbers is Subset of Set of Complex numbers.

$$(a, b) + (c, d) = (a+c, b+d) \quad \text{--- (1)}$$

$$= (c+a, d+b)$$

∵ '+' is commutative in \mathbb{R}

$$= (c, d) + (a, b) \quad \text{--- (2)}$$

iv) Multiplicative Inverse

② Verify the multiplication properties of the complex numbers.

Solution:

i) Closure Property

$$\forall (a, b), (c, d) \in \mathbb{C}$$

$$(a, b) \cdot (c, d) = (ac-bd, ad+bc) \in \mathbb{C}$$

∵ 'x' is closed in \mathbb{R}

ii) Associative Property

For $(a, b), (c, d), (e, f) \in \mathbb{C}$

Consider

$$[(a, b) \cdot (c, d)] \cdot (e, f) = (ac-bd, ad+bc) \cdot (e, f)$$

$$= ((ac-bd)e - (ad+bc)f, (ac-bd)f + (ad+bc)e)$$

$$= (ace - bde - adf - bcf, acf - bdf + ade + bce)$$

Now consider

$$(a, b) \cdot [(c, d) \cdot (e, f)] = (a, b) \cdot ((ce-df, cf+de))$$

$$= (a(ce-df) - b(cf+de), a(cf+de) + b(ce-df))$$

$$= (ace - adf - bcf - bde, acf + ade + bce - bdf)$$

∴ From ① & ②, we get

$$[(a, b) \cdot (c, d)] \cdot (e, f) = (a, b) \cdot [(c, d) \cdot (e, f)]$$

iii) Multiplicative Identity

$\forall (a, b) \in \mathbb{C}$ we have $(1, 0) \in \mathbb{C}$

such that

$$(a, b) \cdot (1, 0) = (a-0, 0+b)$$

$$= (a, b)$$

∴ $(1, 0)$ is multiplicative identity in \mathbb{C} .

$$\forall (a, b) \in \mathbb{C} \text{ there is}$$

$$\frac{1}{(a, b)} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2 - i^2 b^2} = \frac{a-ib}{a^2 - (-1)b^2} = \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - i \frac{b}{a^2+b^2} \in \mathbb{C} \text{ such that}$$

$$(a, b) \cdot \frac{1}{(a, b)} = 1 = 1+i \cdot 0 = (1, 0)$$

∴ (a, b) and $\frac{1}{(a, b)}$ are multiplicative inverse of each other.

v) Commutative Property

$$\forall (a, b), (c, d) \in \mathbb{C}$$

$$(a, b) \cdot (c, d) = (ac-bd, ad+bc)$$

$$= (ca-db, da+cb)$$

∵ 'x' is commutative in \mathbb{R}

$$= (c, d) \cdot (a, b)$$

③ Verify the distributive law of complex numbers.

$$(a, b) \cdot [(c, d) + (e, f)] = (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$$

$$\text{L.H.S.} = (a, b) \cdot [(c, d) + (e, f)]$$

$$= (a, b) \cdot (c+e, d+f)$$

$$= (a(c+e) - b(d+f), a(d+f) + b(c+e))$$

$$= (ac+ae - bd-bf, ad+af + bc+be)$$

$$\text{R.H.S.} = (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$$

$$= (ac-bd, ad+bc) + (ae-bf, af+be)$$

$$= (ac-bd+ae-bf, ad+bc+af+be)$$

∴ From ① & ②

$$\text{L.H.S.} = \text{R.H.S.}$$

④ Simplify the following

i) $i^9 = (i^2)^4 \cdot i = (-1)^4 \cdot i = 1 \cdot i = i$ Ans.

ii) $i^{14} = (i^2)^7 = (-1)^7 = -1$ Ans.

iii) $(-i)^{19} = [(-1) \cdot (i)]^{19} = (-1)^{19} \cdot i^{19} = -i^{19}$
 $= -(i^2)^9 \cdot i = -(-1)^9 \cdot i$
 $= -(-1) \cdot i = i$ Ans.

iv) $(-1)^{-\frac{21}{2}} = \frac{1}{(-1)^{\frac{21}{2}}} = \frac{1}{[(-1)^{\frac{1}{2}}]^{21}} = \frac{1}{i^{21}}$
 $= \frac{1}{(i^2)^{10} \cdot i} = \frac{1}{(-1)^{10} \cdot i} = \frac{1}{1 \cdot i}$
 $= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1}$
 $= -i$ Ans.

⑤ Write in terms of i

i) $\sqrt{-16} = i \cdot 4$ Ans.

ii) $\sqrt{-5} = \sqrt{(-1)(5)} = \sqrt{-1} \sqrt{5} = i\sqrt{5}$ Ans.

iii) $\sqrt{-\frac{16}{25}} = \sqrt{(-1) \cdot \frac{16}{25}} = \sqrt{-1} \sqrt{\frac{16}{25}} = i \frac{4}{5}$
 $= \frac{4}{5} i$ Ans.

iv) $\sqrt{\frac{1}{-4}}$ Method I
 $= \frac{\sqrt{1}}{\sqrt{-4}} = \frac{1}{\sqrt{(-1)(4)}} = \frac{1}{\sqrt{-1} \sqrt{4}} = \frac{1}{i \cdot 2}$
 $= \frac{1}{2i} = \frac{1}{2i} \times \frac{i}{i} = \frac{i}{2i^2} = \frac{i}{2(-1)} = \frac{i}{-2}$
 $= -\frac{i}{2}$ Ans.

⑥ $(7, 9) + (3, -5) = (7+3, 9-5) = (10, 4)$ Ans.

⑦ $(8, -5) - (-7, 4) = (8+7, -5-4)$
 $= (15, -9)$ Ans.

⑧ $(2, 6) \cdot (3, 7)$
 $= (2 \cdot 3 - 6 \cdot 7, 2 \cdot 7 + 6 \cdot 3)$
 $= (6 - 42, 14 + 18)$
 $= (-36, 32)$ Ans.

⑨ $(5, -4) \cdot (-3, -2)$
 $= ((5)(-3) - (-4)(-2), (5)(-2) + (-4)(-3))$
 $= (-15 - 8, -10 + 12) = (-23, 2)$ Ans.

⑩ $(0, 3) \cdot (0, 5)$
 $= ((0)(0) - (3)(5), (0)(5) + (3)(0))$
 $= (0 - 15, 0 + 0) = (-15, 0)$ Ans.

⑪ $(2, 6) \div (3, 7) = \frac{(2, 6)}{(3, 7)}$
 $= \frac{2+6i}{3+7i} = \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i}$
 $= \frac{6-14i+18i-42i^2}{(3)^2 - (7i)^2} = \frac{6+4i-42(-1)}{9-49i^2}$
 $= \frac{6+4i+42}{9-49(-1)} = \frac{48+4i}{9+49} = \frac{48+4i}{58}$
 $= \frac{48}{58} + \frac{4}{58} i = \frac{24}{29} + \frac{2}{29} i$

Ans.

⑫ $(5, -4) \div (-3, -8)$
 $= \frac{(5, -4)}{(-3, -8)} = \frac{5-4i}{-3-8i} = \frac{5-4i}{-3-8i} \times \frac{-3+8i}{-3+8i}$
 $= \frac{-15+40i+12i-32i^2}{(-3)^2 - (8i)^2}$
 $= \frac{-15+52i-32(-1)}{9-64i^2} = \frac{-15+52i+32}{9+64}$
 $= \frac{17+52i}{73} = \frac{17}{73} + \frac{52}{73} i$
 $= (\frac{17}{73}, \frac{52}{73})$ Ans.

⑬ Let the two conjugate complex numbers be $z = x + iy$ and $\bar{z} = x - iy$ where $x, y \in \mathbb{R}$

Sum = $z + \bar{z}$
 $= x + iy + x - iy$
 $= 2x \in \mathbb{R} \quad \because x \in \mathbb{R}$

Product = $z \cdot \bar{z}$

$$= (x+iy)(x-iy)$$

$$= x^2 - i^2 y^2$$

$$= x^2 - (-1)y^2 = x^2 + y^2 \in \mathbb{R}$$

$\because x, y \in \mathbb{R}$

$= (a+2bi)(a-2bi)$ Ans.

(14) i) $(-4, 7)$

Let $z = (-4, 7)$

Multiplicative inverse of $z = \frac{1}{z}$

$$= \frac{1}{(-4, 7)} = \frac{1}{-4+7i} = \frac{1}{-4+7i} \times \frac{-4-7i}{-4-7i}$$

$$= \frac{-4-7i}{(-4)^2 - (7i)^2} = \frac{-4-7i}{16-49i^2}$$

$$= \frac{-4-7i}{16-49(-1)} = \frac{-4-7i}{16+49} = \frac{-4-7i}{65}$$

$$= -\frac{4}{65} - \frac{7}{65}i = \left(-\frac{4}{65}, -\frac{7}{65}\right)$$

Ans.

ii) Let $z = (\sqrt{2}, -\sqrt{5})$

Multiplicative inverse of $z = \frac{1}{z}$

$$= \frac{1}{(\sqrt{2}, -\sqrt{5})} = \frac{1}{\sqrt{2}-\sqrt{5}i}$$

$$= \frac{1}{\sqrt{2}-\sqrt{5}i} \times \frac{\sqrt{2}+\sqrt{5}i}{\sqrt{2}+\sqrt{5}i} = \frac{\sqrt{2}+\sqrt{5}i}{(\sqrt{2})^2 - (\sqrt{5}i)^2}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{2-5i^2} = \frac{\sqrt{2}+\sqrt{5}i}{2-5(-1)} = \frac{\sqrt{2}+\sqrt{5}i}{2+5}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{7} = \frac{\sqrt{2}}{7} + \frac{\sqrt{5}}{7}i$$

$$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$$

Ans.

iii) Let $z = (1, 0)$

Multiplicative inverse of $z = \frac{1}{z}$

$$= \frac{1}{(1, 0)} = \frac{1}{1+i \cdot 0} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$= 1+i \cdot 0 = (1, 0)$$

Ans.

(15) i) $a^2 + 4b^2$

$$= a^2 - (-1)4b^2 = a^2 - i^2 2^2 b^2$$

$$= (a)^2 - (i2b)^2$$

$$= (a+i2b)(a-i2b)$$

ii) $9a^2 + 16b^2$

$$= 9a^2 - (-1)16b^2$$

$$= (3a)^2 - i^2 4^2 b^2 = (3a)^2 - (i4b)^2$$

$$= (3a+i4b)(3a-i4b)$$

$$= (3a+4bi)(3a-4bi)$$

Ans.

iii) $3x^2 + 3y^2$

$$= 3(x^2 + y^2)$$

$$= 3[x^2 - (-1)y^2]$$

$$= 3[(x)^2 - i^2 y^2] = 3[(x)^2 - (iy)^2]$$

$$= 3(x+iy)(x-iy)$$

Ans.

(16) i) $\frac{2-7i}{4+5i} = \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$

$$= \frac{8-10i-28i+35i^2}{(4)^2 - (5i)^2}$$

$$= \frac{8-38i+35(-1)}{16-25i^2} = \frac{8-38i-35}{16+25}$$

$$= \frac{-27-38i}{41} = -\frac{27}{41} - \frac{38i}{41}$$

Ans.

ii) $\frac{(-2+3i)^2}{1+i} = \frac{(-2)^2 + (3i)^2 + 2(-2)(3i)}{1+i}$

$$= \frac{4+9i^2-12i}{1+i} = \frac{4-9-12i}{1+i} = \frac{-5-12i}{1+i}$$

$$= \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{-5+5i-12i+12i^2}{1^2 - i^2} = \frac{-5-7i-12}{1-(-1)}$$

$$= \frac{-17-7i}{2} = -\frac{17}{2} - \frac{7}{2}i$$

Ans.

iii) $\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1-i^2}$

$$= \frac{i-(-1)}{1-(-1)} = \frac{i+1}{2} = \frac{1+i}{2}$$

$$= \frac{1}{2} + \frac{i}{2}$$

Ans.
